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Non-Modal Stability Analysis Of The Zero-Dimensional Model Of TRIGA Mark II Reactor

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ABSTRACT

Non-modal stability theory is a new approach to stability analysis focused on the short-term behaviour of the system following a perturbation. Traditional modal techniques focus on the asymptotic response, missing finite-time instabilities and temporary growth of the parameters. Although widely used in the fluid-dynamics analysis, only seldom applications of this technique exist for nuclear reactors. Building upon previous works done by the author, in the following, the non-modal stability theory is used to study the stability of the zero-dimensional model of the TRIGA Mark II reactor, focusing on the short-term response to perturbations (power, inlet temperature). The primary aim of this work is to show how the widely-used modal stability techniques fail to predict transient growth of the perturbation for asymptotically stable systems, which, if repeated, could potentially lead to additional wear and tear of the system.

1 INTRODUCTION

Nuclear reactors are complex and multi-physics systems, yet their stability analysis still refers to traditional modal techniques. These methods focus only on the asymptotic behaviour of the system following a perturbation, neglecting any short-term response. However, especially for advanced reactor concepts and when fast transients are concerned, the traditional methodologies have started to show their limitations [1]. Along with the asymptotic responses, studies [2] now must consider the short-term behaviour of the perturbed system as well. Stability over finite-time intervals is quite different from the traditional concept of Lyapunov' stability [4]: classic stability theory fails in all those cases for which the asymptotic behaviour is not enough to describe the actual perturbation dynamics [3]. In particular, non-negligible but transient short-term energy amplification may occur even for systems with no predicted long-time disturbance growth. This behaviour implies that energy growth may amplify limited disturbances into larger ones, which may cause the transition from a stable to an unstable state.

Among all techniques, the focus of this paper is the non-modal stability method proposed by Schmidt [5]. This method focuses on the short-term behaviour of the perturbed system over a limited time interval. To the best of the author's knowledge, only seldom works consider non-modal stability analysis to study the dynamics of nuclear reactors. To this end, this paper builds from a preliminary study performed by the author [6] on the zero-dimensional model of a PWR core, presenting the application of the non-modal stability theory on the zero-dimensional model of the whole TRIGA Mark II reactor [7]. The TRIGA (Training Research and Isotope production General Atomics) is a pool-type research reactor with a maximum power of 250 kW.

It presents an asymmetric core configuration, with a secondary cooling system to provide additional cooling through forced convection to avoid excessive increases in the pool temperature. Despite this, natural convection remains the driving force for the flow through the core.

2 NON-MODAL STABILITY ANALYSIS

Non-modal stability focuses on finding the system parameters, initial conditions $\mathbf{y}_0$ and external forcing functions $\mathbf{u}$ (upon which the evolution of the perturbation depends) that result in the maximum amplification of an appropriately chosen quantity, which represents the disturbance magnitude. Indicating the output variables by $\mathbf{y}$, the Lagrange formula allows to obtain the general solution to the initial-value problem with external forcing:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0 + \mathbf{C}\int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t), \tag{1}$$

where $\mathbf{A}$ is the dynamic matrix, $\mathbf{B}$ is the control matrix, $\mathbf{C}$ is the sensors matrix and $\mathbf{D}$ is the direct terms matrix, all taken as constant in time. This general formulation separates the effects of the initial condition and external forcing perturbations, and Equation 1 represents the dynamics of the system. Considering only the response to generic disturbances on the initial conditions, Equation 1 becomes:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0. \tag{2}$$

Non-modal stability analysis aims at identifying the initial condition, parameter and external forcing values whose perturbation results in the maximum amplification of an appropriately chosen quantity. Following hydrodynamic stability theory, this work considers the maximum amplification of energy in a defined time interval. Based on Equation 1 and considering only perturbations on the initial conditions:

$$G(t) = \max_{\mathbf{y}_0} \frac{||\mathbf{y}(t)||^2}{||\mathbf{y}_0||^2} = \max_{\mathbf{y}_0} \frac{||\mathbf{C}e^{\mathbf{A}t}\mathbf{B}\mathbf{y}_0||^2}{||\mathbf{y}_0||^2} = ||\mathbf{C}e^{\mathbf{A}t}\mathbf{B}||^2. \tag{3}$$

This function $G(t)$ describes the maximum energy amplification over all possible initial conditions, and therefore it is independent of the specific initial condition. Given this function, singular value decomposition (SVD) of the matrix exponential at a given time t can retrieve the specific initial perturbation that causes the maximum possible amplification $G(t)$ [8]. Now, traditional linear stability theory characterises the growing of disturbances considering the least stable mode of $\mathbf{A}$, because according to this theory, its specific perturbation governs the system dynamics. This equates to substituting the norm of $G(t)$ with $g(t) = e^{2\lambda_R t}$, where λ_R represents the real part of the least stable eigenvalue of $\mathbf{A}$. Considering the eigenvalue decomposition $\mathbf{A} = \mathbf{W}\mathbf{\Lambda}\mathbf{W}^{-1}$, nothing about the eigenfunction of $\mathbf{A}$, contained in $\mathbf{W}$, is taken into account:

$$G(t) = ||\mathbf{C}e^{\mathbf{A}t}\mathbf{B}||^2 = ||\mathbf{C}\mathbf{W}e^{t\mathbf{\Lambda}}\mathbf{W}^{-1}\mathbf{B}||^2 \tag{4}$$

Spectral analysis is the analysis of the real part of the least stable eigenvalue, and it describes the time-evolution of the norm of the matrix exponential only for unitary matrices $\mathbf{W}$ because the similarity transformation does not change its value. However, many operators are non-normal, and their eigenfunctions are non-orthogonal. As such, the study of λ_R could not fully capture the general disturbance behaviour in time. In addition, the dynamic behaviour of the least stable eigenvalue may not be significant for the one at a finite time.

Systems characterised by non-normal matrices may experience a short but large energy amplification following a perturbation of the initial condition. If the dynamic matrix $\mathbf{A}$ eigenvectors form a non-orthogonal set, a transient increase of energy occurs even when the least stable eigenvalue lie left of the imaginary axis (and hence predicts asymptotic decay). To capture the behaviour of the infinitesimal disturbances, the norm of the matrix exponential $G(t)$ must be taken into account because it contains the full dynamical information without any restrictive assumptions. For the behaviour at infinitesimal times, a different quantity than the eigenvalues is needed. Considering a Taylor expansion of the matrix exponential about $t = 0^+$, the following can be obtained [8]:

$$\max_{y} \frac{1}{||\mathbf{y}||^2} \frac{d||\mathbf{y}||^2}{dt} = \lambda_R(\mathbf{Q}^H + \mathbf{Q}), \tag{5}$$

where $\mathbf{Q} = \frac{1}{2}\mathbf{CAB}$. The quantity $\lambda_R(\mathbf{Q}^H + \mathbf{Q})$ is the numerical abscissa, which represents the maximum extension of the numerical range of the right half-plane. The numerical abscissa determines the energy increase for $t = 0^+$. To further investigate the maximum short-time energy growth, the ε-pseudo-spectrum [8] is used. They are the regions in the complex plane, parametrised by ε, where the resolvent norm $||\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}||$ is larger than $1/\varepsilon$. Alternatively, the ε-pseudo-eigenvalues are the eigenvalues of the operator $\mathbf{A} + \mathbf{E}$, where $\mathbf{E}$ is the disturbance of norm ε. As ε goes to zero, the regular spectrum can be retrieved [8].

3 MODEL DESCRIPTION

The present work adopts the zero-dimensional model of the TRIGA Mark II reactor described and validated in [7]. To accurately simulate the complete dynamic behaviour of the system, this model considers both core and pool physics. In particular, the neutronic description of the reactor core includes, other than the six-group point reactor kinetic equation, the dynamic evolution of the two most significant neutron poisons, Xenon $^{135}_{54}\mathrm{Xe}$ and Samarium $^{149}_{62}\mathrm{Sm}$ (along with their precursors, respectively Iodine $^{135}\mathrm{I}$ and Promethium $^{149}\mathrm{Pm}$. The reactivity evolution equation now includes the fuel and coolant temperature feedbacks, the coolant density feedback and the neutron poisons feedbacks.

From the thermal-hydraulics aspect, the model considers a two-region (fuel and coolant) approach, based on the average temperature, a total heat exchange coefficient between fuel and coolant derived from the overall thermal resistance between the two regions, and the Ruiz correlation [9] for the coolant convective heat transfer. The overall mass flow rate within the core derives from the momentum balance on the coolant in the vertical direction between friction and buoyancy forces (under the Boussinesq approximation for buoyancy-driven flows) and neglecting all concentrated pressure losses. Finally, the model also includes the evolution of the primary and secondary pool temperatures through energy balances. Figure 1 reports the Simulink schematic for the model. The resulting system of sixteen ordinary differential equations can be written adopting the state-space representation:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} & (6) \\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} & (7) \end{cases}$$

where $\mathbf{x}$ is the state of the system, $\mathbf{u}$ the system input and $\mathbf{y}$ the system output:

$$\begin{cases} \mathbf{x} = [\psi\ \eta_i\ T_f\ T_c\ \Gamma\ \mathrm{I}\ \mathrm{Xe}\ \mathrm{Pm}\ \mathrm{Sm}\ T_p\ T_r]' & (8) \\ \mathbf{u} = [\Delta h_{rod}\ P_0\ T_{aq}] & (9) \end{cases}$$

where ψ is the normalised neutron density, η_i is the normalised i-th precursor concentration, T_f is the average fuel temperature, T_c is the average coolant temperature, Γ is the coolant mass flow rate, T_p is the average pool temperature, T_r is the average reservoir temperature, Δh_{rod} is the input rod insertion, P_0 is the desired steady-state power and T_{aq} is the external aqueduct temperature, which acts as input for the two pools. In general, stability analysis refers the dynamic matrix $\mathbf{A}$, which for the present model has the form:

$$\begin{bmatrix}
\frac{\rho_0-\beta}{\Lambda} & \frac{\beta_i}{\Lambda} & \frac{\alpha_f}{\Lambda} & \frac{\alpha_c}{\Lambda} & 0 & 0 & -\frac{\alpha_{Xe}}{\Lambda} & 0 & -\frac{\alpha_{Sm}}{\Lambda} & 0 & 0 \\
-\lambda_i & \lambda_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{P_0}{\tau_f K} & 0 & -\frac{1}{\tau_f} & \frac{1}{/}\tau_f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\tau_c} & -\left(\frac{1}{\tau_c}+\frac{2C_c\Gamma_0}{\tau_c K}\right) & \frac{2*C_c(T_{c,0}-T_{p,0})}{\tau_c K} & 0 & 0 & 0 & 0 & \frac{2C_c\Gamma_0}{\tau_c K} & 0 \\
0 & 0 & 0 & -2\rho_0\nu\mathbf{g}A_{flow} & -\frac{2\Gamma_0 A_{flow} f_d}{h_{core}} & 0 & 0 & 0 & 0 & 2\rho_0\nu\mathbf{g}A_{flow} & 0 \\
y_I\Sigma_{fiss}\phi_0 & 0 & 0 & 0 & 0 & -\lambda_I & 0 & 0 & 0 & 0 & 0 \\
\phi_0(y_{Xe}\Sigma_{fiss}-\Sigma_{Xe}) & 0 & 0 & 0 & 0 & \lambda_I & -\lambda_{Xe}-\Sigma_{Xe}\phi_0 & 0 & 0 & 0 & 0 \\
y_{Pm}\Sigma_{fiss}\phi_0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_{Pm} & 0 & 0 & 0 \\
-\Sigma_{Sm}*\phi_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_{Pm} & -\Sigma_{Sm}*\phi_0 & 0 \\
0 & 0 & 0 & \frac{\Gamma_0}{M_p} & \frac{T_{c,0}-T_{p,0}}{M_p} & 0 & 0 & 0 & 0 & -\frac{\Gamma_0+\dot{m}_1}{M_p} & \frac{\dot{m}_1}{M_p} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\dot{m}_1}{M_r} & -\frac{\dot{m}_1+\dot{m}_3}{M_r}
\end{bmatrix} \tag{10}$$

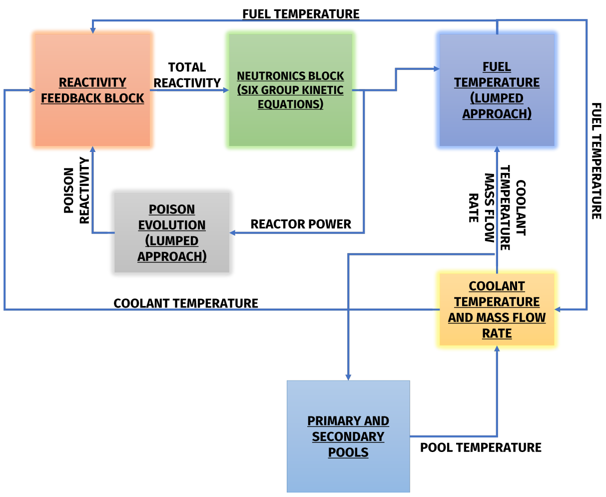
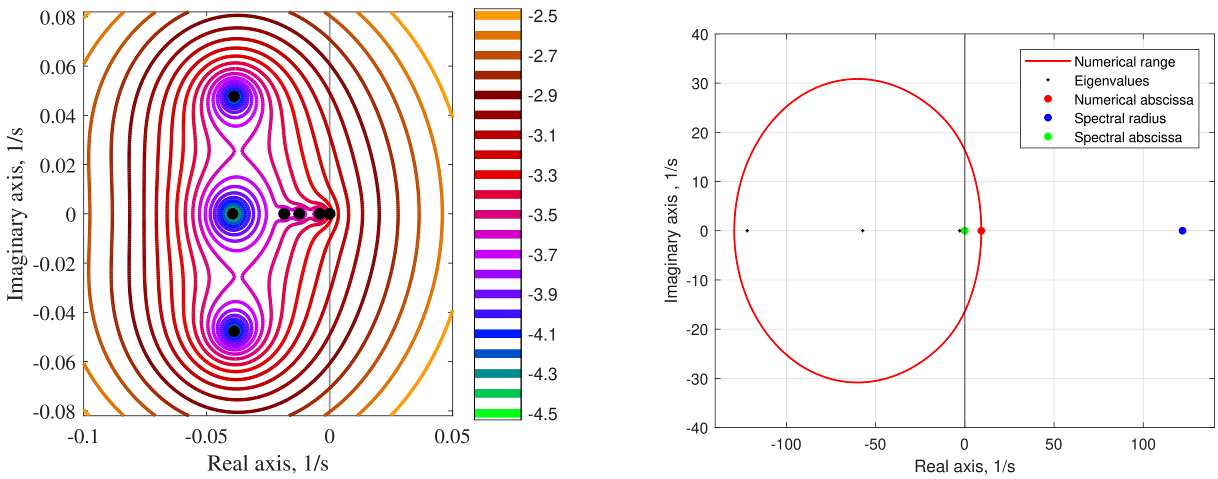


Figure 1: Simulink representation of the reactor model.

4 RESULTS AND DISCUSSION

The analysis considers small, generic perturbations of the dynamic matrix $\mathbf{A}$. For linear stability theory, the eigenvalues of this matrix, in particular the largest one, can fully describe stability. However, this is strictly true only for matrices with condition number (which measures the sensitivity to perturbations in the input data) lower than 1. In all other cases, the least stable eigenvalue only predicts the asymptotic stability and does not give any information about the short-term response of the matrix. This limit holds in particular for non-orthogonal matrices, such as the one describing the TRIGA Mark II reactor (Matrix 10). Indeed, as seen in Figure 2b, all eigenvalues are located in the left half-plane, and according to linear, they predict asymptotic decay governed by the least stable one. However, the numerical range crosses into the right half-plane, and some energy growth can occur over finite-time intervals. Table 1 further proves this by showing the most relevant parameters for the unperturbed non-dimensional dynamic matrix. In particular, the higher the departure from normality value, the farthest from orthogonality is the matrix. The matrix $\mathbf{A}$ is thus non-orthogonal, and the temporal behaviour of $G(t)$ differs from the one of $g(t)$.



(a) (b)

Figure 2: (a) Pseudo-spectra of the dynamic matrix $\mathbf{A}$ for different value of the perturbation (steady-state power equal to 250 kW). The colour bar shows the norm of the perturbation on a logarithmic scale, expressed as $\log \varepsilon$; (b) Numerical range, spectral radius, spectral abscissa and numerical abscissa of the dynamic matrix $\mathbf{A}$.

Table 1: Main parameters for the unperturbed non-dimensional dynamic matrix $\mathbf{A}$.

Departure from normality	78.7621
Condition number	194.2449
Numerical abscissa	9.2117
Spectral abscissa	-1.89e-07
Spectral radius	122.0216

Figure 2a shows the contour levels of the norm for generic values of the perturbation ε, referring only to the largest eigenvalues as they are the ones whose variations most impact stability. The pseudo spectrum confirms that, even for small perturbations, the contour levels

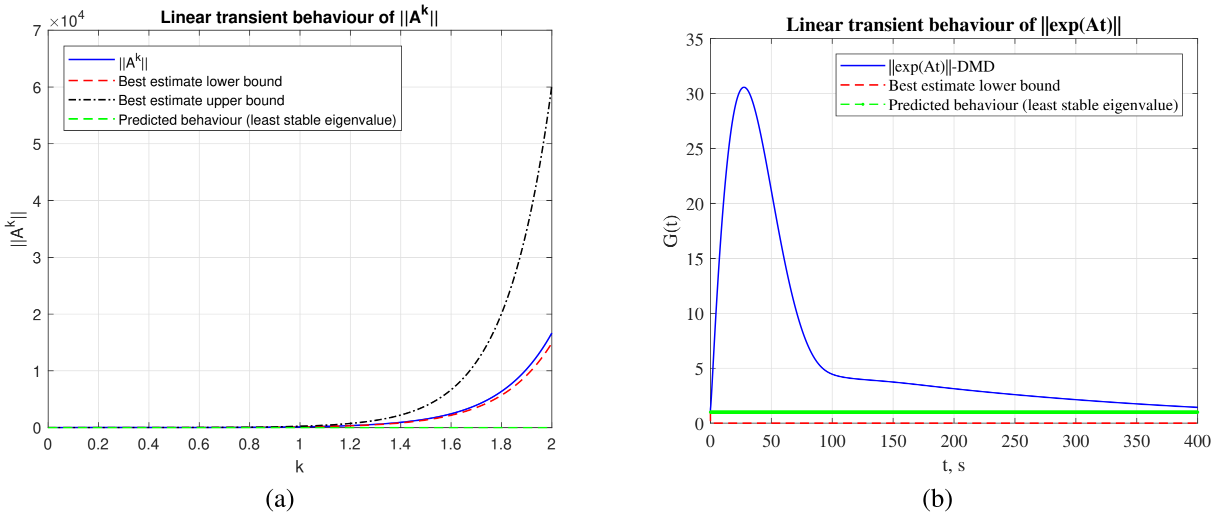


Figure 3: Linear transient growth of the non-dimensional matrix power $||\mathbf{A}^k||$ (with the best estimate upper and lower bounds), and of the perturbation energy $||\exp \mathbf{A}t||$ with respect to time . The dashed line shows the behaviour predicted by the least stable eigenvalue.

cross into the right half-plane, indicating the possibility of short-term energy growth at the beginning of the perturbation transient.

Figure 3 confirms this by showing the behaviour of the non-dimensional matrix power $||\mathbf{A}^k||$ and the perturbation energy $G(t)$. As the spectral radius, the largest absolute value of the eigenvalues of the matrix $\mathbf{A}$ is larger than 1, the matrix power does not converge as $k \to \infty$, indicating amplification of the perturbation (Figure 3a). The best estimate upper and lower bounds are:

$$\text{Lower bound: } \max_{1\leq k\leq K} ||\mathbf{A}^k|| \geq \frac{\rho^K}{1+\frac{\rho^K-1}{\rho||\rho-\mathbf{A}||^{-1}-1}} \tag{11}$$

$$\text{Upper bound: } \min_{1\leq k\leq K} ||\mathbf{A}^k|| \leq ||\mathbf{A}||^K \tag{12}$$

Indeed, the perturbation energy $G(t)$ (Figure 3b), shows initial energy growth with a peak around 25 seconds at the beginning of the transient. $G(t)$ then decreases, initially sharply until 80 seconds, then slower until, after roughly 450 seconds, the perturbation energy decay follows the one predicted by the least stable eigenvalue. Interestingly, for the TRIGA Mark II reactor, short-term energy growth occurs over a reasonably long time interval. In terms of stability and control, this is an advantage compared to the fastest transients because it gives time to the operator to monitor the quantities of interest during this initial phase of the perturbation, further confirming good stability properties and inherent safety of this kind of reactor.

Now the response of the dynamic matrix $\mathbf{A}$ to specific perturbations in the input parameters are considered. In particular, this work refers to two inputs parameters: the desired steady-state power (kW) and the pool temperature (°C). The former has been chosen to study the stability of the system under non-nominal power conditions, for example, during pulse mode operation (in which the reactor has peak power around 250 MW); the latter, instead, considers the situation in which the starting temperature of the pool is higher than the ambient one. The investigated parameter values are:

$$\begin{cases} T_{p,0} = [20\ 50\ 100\ 150\ 200\ 250] & (13) \\ \varepsilon_{TP} = [0.003\ 0.0061\ 0.01\ 0.013\ 0.0155] & (14) \\ P_0 = [250\ 1000\ 250000] & (15) \\ \varepsilon_P = [0.2624\ 443.2071] & (16) \end{cases}$$

where the vector ε represents the magnitude of the perturbation obtained by varying, respectively, the pool temperature and the steady-state power compared to the unperturbed matrix.

Figure 4a shows the pseudo-spectrum of the dynamic matrix $\mathbf{A}$ for variations of $T_{p,0}$ (Equation 14). Even with small perturbations, the contour levels cross into the right half-plane, indicating transient energy growth. Figure 4b show the linear transient increase of energy $G(t)$ following perturbations on the pool initial temperature. The results again confirm the robustness of the TRIGA Mark II reactor against perturbations on this parameter: even in the unlikely case of initial temperature well above the water saturation one, transient growth of energy is followed by a decrease according to the least stable eigenvalue. As expected, as the magnitude of the perturbation increases, the peak of $G(t)$ shifts slightly towards the right, indicating a longer duration of the energy growth phase, and conversely, the larger the magnitude of the perturbation, the higher the peak.

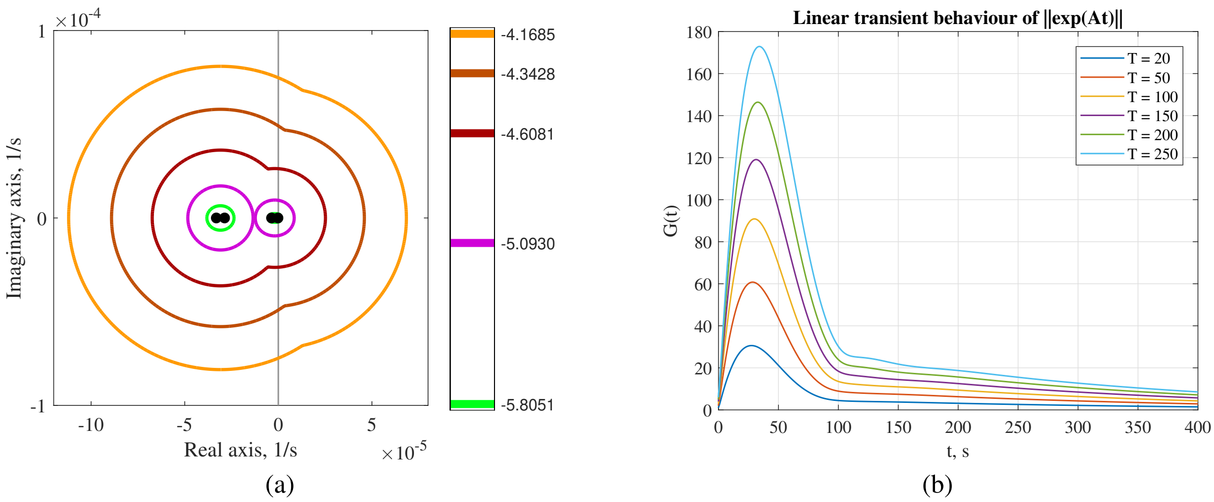
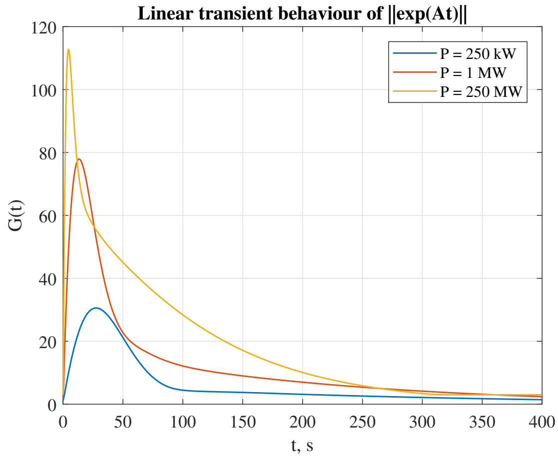


Figure 4: (a) Pseudo-spectra of the dynamic matrix $\mathbf{A}$ for different value of the perturbation on $T_{p,0}$. The colour bar reports $\log \varepsilon$. (b) Linear transient growth of the perturbation energy $|| \exp \mathbf{A}t||$ with respect to time for perturbations on the input parameters $T_{p,0}$.

Figure 5a show the linear transient increase of energy $G(t)$ following large-scale perturbations on the steady-state power. In particular, the maximum value allowed during pulse-mode operation is considered (250 MW). Even in this latter case, a decrease of energy according to the least stable eigenvalue follows transient growth. Compared to variations in the pool temperature, for perturbations in the steady-state power $G(t)$ is quite different in the three cases: in particular, the higher P_0, the narrower and shorter the peak is. Indeed, as the perturbation increases, the maximum value of $G(t)$ shifts to lower times and the decrease to the asymptotic value is slower. As the growth phenomenon is shorter but more significant, this indicates less time for monitoring the quantities of interest during this time interval, making detecting transient peak values harder.



(a)

Figure 5: Linear transient growth of the perturbation energy $|| \exp \mathbf{A}t||$ with respect to time for perturbations on the input parameters P_0.

5 CONCLUSIONS

This paper shows a preliminary stability analysis of the zero-dimensional model of the TRIGA Mark II reactor to perturbations of the input variables (pool temperature and steady-state power), using the non-modal stability methodology to focus on the short-term behaviour of the system following the disturbance (instead of relying on asymptotic stability only, as commonly done for nuclear reactors). Even considering limited perturbations in the dynamic matrix $\mathbf{A}$ (such as those caused by variations in the pool temperature), non-modal stability analysis observes transient growth of the amplification energy $G(t)$: this means that the sign of the least-stable eigenvalue is significant only for asymptotic stability, and cannot give any information about the short-term behaviour. In general, the analysis confirms the robustness properties of the TRIGA Mark II reactor: even considering a steady-state power of 250 MW (the maximum value reached during pulse-mode operation), $G(t)$ still decays to the asymptotic value, proving the overall stability of the system and confirming the usefulness of this method for future applications to Generation-IV systems.

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